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2-GROUPS OF NORMAL RANK 2 FOR WHICH THE FRATTINI SUBGROUP HAS RANK 3(1)

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Abstract. All finite 2-groups G with the following property are classified:
Property. The Frattini subgroup of G contains an abelian subgroup of rank 3, but G contains no normal abelian subgroup of rank 3.
The method of classification involves showing that if G is such a group, then G contains a normal abelian subgroup W isomorphic to $Z_4 \times Z_4$, and that the centralizer C of W in G has an uncomplicated structure. The groups with the above property are then constructed as extensions of C.

Introduction. In recent years there has been interest in the class of finite 2-groups which contain no normal abelian subgroup of rank three (cf. [3]).

The 2-groups which contain no normal abelian subgroup of rank two have been known for some time. In classifying those 2-groups which have no normal abelian subgroup of rank two, one may first show that if G is such a group, then the Frattini subgroup of G contains no elementary abelian subgroup of order 4 [2, Satz III, 7.5]. Next one shows that G has a cyclic subgroup of index 2 [2, Satz III, 7.6], and then one may easily construct the 2-groups which have no normal elementary abelian subgroup of order 4 [2, Satz I, 14.9b].

Therefore, as a first step in the classification of 2-groups G which contain no normal abelian subgroup of rank three, it seems reasonable to consider groups G with the following property:

($*$) G is a 2-group and $\Phi(G)$, the Frattini subgroup of G, contains an elementary abelian subgroup of order 8, but G contains no normal elementary abelian subgroup of order 8.

In this paper we classify all groups with property ($*$). Such a group will be called a ($*$)-group.

The paper is divided into four sections. In the first section we discuss the general structure of 2-groups satisfying ($*$). We show that if G is such a group, then $\Phi(G)$ contains a subgroup W which is normal in G, and W is isomorphic to the direct

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product of two cyclic groups of order 4. We then show that C, the centralizer of W in G, is isomorphic to one of the groups

$$C(n, n, \varepsilon) = \langle a, b | a^{2^n} = b^{2^n} = 1, [a, b] = a^{\varepsilon 2^{n-1}} \text{ for } \varepsilon \in Z_2 \text{ and } n \geqq 2\rangle,$$

or

$$C(n+1, n, \varepsilon) = \langle a, b | a^{2^{n+1}} = b^{2^n} = 1, [a, b] = a^{\varepsilon 2^n} \text{ for } \varepsilon \in Z_2 \text{ and } n \geqq 2\rangle.$$

The 2-groups satisfying (*) can then be constructed as extensions of C by automorphisms of C which are extensions of automorphisms of W. In §§2, 3, and 4 we construct all 2-groups G satisfying (*) in which $c_G(W)=W$, $|c_G(W)/W|=2$, and $|c_G(W)/W|>2$, respectively. The work in these sections involves quite a bit of calculation, but I feel that it is justified since knowledge of the exact structure of these groups may be quite useful in further research.

We summarize our result in the following theorem.

MAIN THEOREM. *Let G be a finite 2-group having the following property*:

(*) *$\Phi(G)$ contains an elementary abelian subgroup of order* 8, *but G contains no normal elementary abelian subgroup of order* 8.

Then G is isomorphic to one of the following groups:

I. *The groups G for which $W=c_G(W)$*:

1. $S(i,j)=\langle k, s \mid k^4=s^8=1, k^s=ks^{2j}u^i$ *where* $i\equiv j\equiv 1 \pmod 2$, $u^4=1$, $u^s=us^4$, $u^k=u^{-1}s^2\rangle$.
2. $H=\langle g, k \mid g^2=k^4=1, k^g=k^{-1}u, u^4=1, u^g=u^k=u^{-1}y, y^4=[u, y]=1, y^k=u^2y\rangle$.
3. $T(\alpha, \beta, i, j)=\langle t, k \mid t^{16}=k^4=1, k^t=k^{-1}t^{2+4\beta}u^\alpha$ *where* $\alpha\equiv\beta\equiv 1 \pmod 2$, $u^4=1$, $k^{t^2}=kt^{4j}u^i$ *where* $i\equiv j\equiv 1 \pmod 2$, $u^t=ut^4$, $u^k=u^{-1}t^4\rangle$.
4. $\tilde{S}(i,j)=\langle r, s \mid r^4=s^8=1, r^s=rs^{2j}u^i$ *where* $i\equiv j\equiv 1 \pmod 2$, $u^4=1$, $u^r=us^2$, $u^s=us^4\rangle$.
5. $\hat{S}(i,j)=\langle r, s \mid r^8=1, r^4=s^4, r^s=rs^{2j}u^i$ *where* $i\equiv j\equiv 1 \pmod 2$, $u^4=1$, $u^r=us^2$, $u^s=us^4\rangle$.
6. $\tilde{H}(i,j)=\langle g, r \mid r^4=g^2=1, r^g=r^{-1}u^iy^j$ *where* $i\equiv j\equiv 1 \pmod 2$, $u^4=y^4=[u, y]=1$, $u^g=u^{-1}y$, $u^r=uy$, $y^r=u^2y\rangle$.
7. $\hat{H}(\delta, i, j)=\langle r, g \mid r^8=g^2=1, r^g=r^{-1}u^iy^j$ *where* $i, j\in Z_4$, $y^2=r^4$, $u^4=[u, y]=1$, $u^r=uy$, $y^r=u^2y$, $u^g=u^{-1}y^\delta$ *where* $\delta=\pm 1\rangle$.
8. $\tilde{X}(m, n, i, j, \varepsilon)=\langle g, \tilde{S}(i,j) \mid g^2=1, r^g=r^{-1}u^ms^{2n}$ *where* $m\equiv n\equiv 1 \pmod 2$, $s^g=s^{1+4\varepsilon}$ *where* $\varepsilon=\pm 1\rangle$.
9. $\hat{X}(m, n, i, j, \varepsilon)=\langle g, \hat{S}(i,j) \mid g^2=1, r^g=r^{-1}u^ms^{2n}$ *where* $m\equiv n\equiv 1 \pmod 2$, $s^g=s^{1+4\varepsilon}$ *where* $\varepsilon=\pm 1\rangle$.

II. *The groups for which* $|c_G(W)/W|=2$:

10. $K_\alpha=\langle a, k \mid a^8=k^4=1, a^k=au, u^4=[a, u]=1, u^k=u^{-1}a^{2+4\alpha}$ *where* $\alpha\in Z_2\rangle$.
11. $\hat{K}=\langle a, k \mid a^8=1, a^4=k^4, a^k=au, u^4=[a, u]=1, u^k=u^{-1}a^2\rangle$.
12. $H_0(i, j, \beta)=\langle g, K_0 \mid g^2=1, k^g=k^{-1}u^iy^{4j}, i\in Z_4, j\in Z_2, u^g=a^2u^{-1}, a^g=a^{1+4\beta}$ *where* $\beta\in Z_2\rangle$.
13. $\hat{H}_0(i, j, \beta)=\langle g, \hat{K} \mid g^2=1, k^g=k^{-1}u^ia^{4j}$ *where* $i\in Z_4, j\in Z_2$, $u^g=a^2u^{-1}$, $a^8=a^{1+4\beta}$ *where* $\beta\in Z_2\rangle$.

14. $H_1(i, j, \beta) = \langle g, K_1 \mid g^2 = 1, k^g = k^{-1}u^i a^{4j}$ *where* $i \in Z_4, j \in Z_2, u^g = u^{-1}a^{-2}, a^g = a^{1+4\beta}$ *where* $\beta \in Z_2\rangle$.
15. $H(i, j, \alpha, a) = \langle g, K_\alpha \mid g^2 = a, k^g = k^{-1}u^i a^{-1+4j}$ *where* $i \in Z_4, j \in Z_2, u^g = u^{-1}a^{2+4\alpha}$ *where* $\alpha \in Z_2\rangle$.
16. $\hat{H}(i, j, a) = \langle g, \hat{K} \mid g^2 = a, k^g = k^{-1}u^i a^{-1+4j}$ *where* $i \in Z_4, j \in Z_2, u^g = u^{-1}a^2\rangle$.
17. $R_\alpha = \langle r, a \mid r^4 = a^8 = 1, a^r = ua, u^4 = [u, a] = 1, u^r = ua^{2+4\alpha}$ *where* $\alpha \in Z_2\rangle$.
18. $\hat{R}_\alpha = \langle r, a \mid r^8 = 1, r^4 = a^4, a^r = ua, u^4 = [u, a] = 1, u^r = ua^{2+4\alpha}$ *where* $\alpha \in Z_2\rangle$.

III. *The groups for which* $|c_G(W)/W| > 2$:

19. $K(n, n, \varepsilon, w) = \langle b, k \mid k^4 = b^{2^n} = 1$, *where* $n \geqq 3, b^k = b^{-1}a, a^{2^n} = 1, [a, b] = a^{\varepsilon 2^{n-1}}$ *where* $\varepsilon \in Z_2, a^k = b^{-2}aw$, *where* $w \in \langle a^{2^{n-1}}, b^{2^{n-1}}\rangle\rangle$.
20. $\hat{K}(n, n, \varepsilon, w) = \langle b, k \mid b^{2^n} = k^8 = 1$ *where* $n \geqq 3, b^k = b^{-1}a, a^{2^{n-1}} = k^4, [a, b] = a^{\varepsilon 2^{n-1}}$ *where* $\varepsilon \in Z_2, a^k = b^{-2}aw$ *where* $w \in \langle a^{2^{n-1}}, b^{2^{n-1}}\rangle\rangle$, *but not* $\varepsilon \equiv 0 \pmod 2$ *and* $w = 1$.
21. $K(n+1, n, \varepsilon, w) = \langle k, a \mid k^4 = a^{2^{n+1}} = 1$ *where* $n \geqq 3, a^k = b^{-1}a, b^{2^n} = 1, [a, b] = a^{\varepsilon 2^n}$ *where* $\varepsilon \in Z_2, b^k = b^{-1}a^2w$ *where* $w \in \langle a^{2^n}, b^{2^{n-1}}\rangle\rangle$.
22. $\hat{K}(n+1, n, \varepsilon, w) = \langle k, a \mid a^{2^{n+1}} = 1, n \geqq 3, k^4 = a^{2^n}, a^k = b^{-1}a, b^{2^n} = 1, [a, b] = a^{\varepsilon 2^n}$ *where* $\varepsilon \in Z_2, b^k = b^{-1}a^2w$ *where* $w \in \langle a^{2^n}, b^{2^{n-1}}\rangle\rangle$, *for* $\varepsilon \equiv 0 \pmod 2$ *and* $w = a^{2^n}$, *or* $\varepsilon \equiv 1 \pmod 2$ *and any choice of* w.
23. *Certain extensions of the groups* 20–22 *of degree two* (*see* §4.4).

Notations.

$Z(G)$, the center of G.

$\Phi(G)$, the Frattini subgroup of G.

$\Omega_k(G)$, the subgroup of G generated by all elements of order 2^k.

$\mho^k(G)$, the subgroup of G generated by all 2^k powers of elements of G.

Aut (G), the automorphism group of G.

$c_G(Y)$, the centralizer in G of Y.

$N_G(Y)$, the normalizer of Y in G.

$A_G(Y)$, the subgroup of Aut (Y) isomorphic to $N_G(Y)/c_G(Y)$.

G-normal, a subgroup H of G is G-normal if $H \triangleleft G$.

E_{2^n}, an elementary abelian subgroup of order 2^n.

Z_n, the integers modulo n.

C_n, a cyclic group of order n.

Four group, the noncyclic group of order 4.

1. **General properties of $(*)$-groups.** Let G be a $(*)$-group. We now show that all $(*)$-groups share certain structural properties.

1.1. *$Z(G) \cap \Phi(G)$ is cyclic.*

Proof. If $Z(G) \cap \Phi(G)$ is not cyclic, then $W_0 = \Omega_1(Z(G) \cap \Phi(G))$ is a four group. Let Y be a normal subgroup of G of order 8 such that $W_0 < Y \leqq \Phi(G)$. Then $YC_4 \simeq \times C_2$. Since the group of automorphisms $A_G(Y)$ acts trivially on W_0, $A_G(Y)$

is elementary abelian. Hence, $c_G(Y)$ contains $\Phi(G)$. Now since $\Phi G\backslash Y$ contains an involution, we see that $\Phi G/W_0$ is not cyclic and so [2, III, 7.5] we see that $\Phi(G)$ contains a G-normal subgroup W such that W/W_0 is a four group. Since $W_0 \leqq Z(G)$, $A_G(W)$ is elementary abelian. So $\Phi(G)$ centralizes W. Now we may apply a theorem of Alperin [1] to show that the elements of order 2 which centralize W lie in W. This contradicts the fact that $\Phi(G)\backslash Y$ (hence, $\Phi(G)\backslash W$) contains an involution. So $\Phi(G) \cap Z(G)$ is cyclic.

1.2. *Let* $W \simeq \langle u, y \mid u^4 = y^4 = [u, y] = 1\rangle$ *and let* A *be a Sylow* 2-*subgroup of the automorphism group of* W. *Then*

$$A = \langle \lambda, \sigma, \rho \mid \lambda^4 = \sigma^4 = \rho^2 = 1, [\sigma, \lambda] = \sigma^2\lambda^2, [\rho, \lambda] = \sigma^2, [\sigma, \rho] = \sigma^2, [\sigma^2, \lambda] = 1\rangle$$

where the actions of the automorphisms of A *are given by*

	λ	σ	ρ	σ^2	$\lambda^2 = \eta$	$\sigma^2\eta$
u	$u^{-1}y$	uy	u^{-1}	uy^2	u^{-1}	$u^{-1}y^2$
y	u^2y	y	y	y	y^{-1}	y^{-1}

In addition, $Z(A) = \Phi(A) = \langle \sigma^2, \eta\rangle$ *and* $|A| = 2^5$.

Proof. See [3, §2, Lemma 1, (ii)].

1.3. $\Phi(G)$ *contains no normal cyclic subgroup of order* >2.

Proof. Assume $\langle y\rangle$ is a G-normal cyclic subgroup of $\Phi(G)$ of order 4. Then y^2 is an involution in $\Phi(G) \cap Z(G)$, and since $|G/c_G(\langle y\rangle)| \leqq 2$, we have $c_G(\langle y\rangle) \geqq \Phi(G)$. Now $\Phi(G)/\langle y^2\rangle$ is not cyclic so [2, III, 7.5] we see that $\Phi(G)$ contains a G-normal subgroup Y such that $\langle y\rangle \leqq Y$ and $Y/\langle y^2\rangle$ is a four group. Thus, $Y \simeq C_4 \times C_2$ and since G normalizes $\langle y\rangle$, we see that the group of automorphisms $A_G(Y)$ is elementary abelian of order 2 or 4. Thus, $Y \leqq Z(\Phi(G))$.

Let $W_0 = \Omega_1(Y)$. Since $\Phi(G)\backslash Y$ contains an involution, $\Phi(G)/W_0$ is not cyclic and so [2, III, 7.5] $\Phi(G)$ contains a G-normal subgroup W such that $W \geqq Y$ and $W \simeq C_4 \times C_4$.

Let $C = c_G(W)$. Then $\Omega_2(C) = W$ [1]. Thus, we can find an involution $v \in \Phi(G)\backslash C \cap \Phi(G)$. Now the automorphism induced by v on W acts trivially on the subgroups $\langle y\rangle$ and W_0. By a simple calculation, it can be seen that the automorphism induced by v lies in the center of $A_G(W)$. So, for all $g \in G$, the action of v^g is the same as the action of v on W. Thus, $vv^g = v^{-1}v^g$ acts trivially on W, so $vv^g \in C$.

Now we consider the order of vv^g. If $(vv^g)^2 = 1$ for all $g \in G$, then $vv^g \in W_0$ (since $\Omega_2(C) = W$) and so $\langle v, W_0\rangle \lhd G$, a contradiction since G has no normal elementary abelian subgroup of order 8.

Thus, the order of vv^g is at least 4 and so, since $(vv^g)^v = (vv^g)^{-1}$, we see that v inverts some element of $W\backslash W_0$. Since v acts as an element of $\Phi(A_G(W))$, we see, by

1.2, that v acts as either η or $\eta\sigma^2$ (σ^2 inverts no element of $W\backslash W_0$). Thus, we have two situations: either the automorphism induced by v is the square of some element in $A_G(W)$ or $\Phi(A_G(W))=\Phi(A)$. In either case, $A_G(W)$ contains an element (λ or $\rho\lambda$) which does not fix the group $\langle y\rangle$, a contradiction.

1.4. $\Phi(G)$ *contains a G-normal subgroup* $W\simeq C_4\times C_4$.

Proof. Since $\Phi(G)$ is not cyclic $\Phi(G)$ contains a G-normal four subgroup W_0 and $W_0\leqq Z(\Phi(G))$. By 1.1 and 1.3 we see that $\langle z\rangle=Z(G)\cap\Phi(G)$ is a cyclic group of order 2. Since $\Phi(G/\langle z\rangle)$ is not cyclic, $\Phi(G)$ contains a G-normal subgroup Y such that $Y\geqq W_0$ and $Y/\langle z\rangle$ is a four subgroup. Thus, $Y\simeq C_4\times C_2$. If $c_{\Phi(G)}(Y)/W_0$ is not cyclic, then $c_{\Phi(G)}(Y)$ contains a G-normal subgroup W such that $W\geqq Y$ and W/W_0 is a four group. Thus, $W\simeq C_4\times C_4$.

So we must show that $c_{\Phi(G)}(Y)/W_0$ is not cyclic. If $c_{\Phi(G)}(Y)/W_0$ is cyclic, then $c_{\Phi(G)}(Y)$ is abelian ($Y\leqq Z(c_{\Phi(G)}(Y))$) and so must have two generators (otherwise $\Omega_1(c_{\Phi(G)}(Y))$ is a G-normal E_8). Thus, $c_{\Phi(G)}(Y)=\langle a, w \mid a^{2^n}=w^2=1, n\geqq 2, [a,w]=1\rangle$. If $n>2$, then $\mho^1(c_{\Phi(G)}(Y))$ is a G-normal cyclic subgroup of $\Phi(G)$ of order greater than 2 which is impossible by 1.3. Thus, $n=2$, i.e. $Y=c_{\Phi(G)}(Y)$.

A Sylow 2-subgroup of Aut (Y) is dihedral of order 8. So $\Phi(A_G(Y))$ has order 2. Thus, $\Phi(G/c_G(Y))=\Phi(G)c_G(Y)/c_G(Y)\simeq\Phi(G)/(\Phi(G)\cap c_G(Y))$. So $|\Phi(G)/c_{\Phi(G)}(Y)|=2$ and $\Phi G\backslash c_{\Phi(G)}(Y)$ contains an involution v, and $v\equiv k^2 \pmod Y$, for some k in G.

Let $g\in G$; then $k^g=k^ia^jw^n$ where $i=\pm 1$ and $j\in Z_4$, $n\in Z_2$. So, $(k^2)^g=(k^ia^jw^n)^2=k^{2i}(a^jw^n)^{k^i}(a^jw^n)$. Hence, $(k^g)^2\equiv k^2 \pmod{W_0}$, i.e. $\langle v, W_0\rangle$ is a G-normal E_8, a contradiction.

So $c_{\Phi(G)}(Y)/W_0$ is not cyclic, hence, G contains a normal subgroup $W\simeq C_4\times C_4$.

1.4.1. *Notation.* Let G be a (*)-group. Then W will denote a G-normal subgroup of $\Phi(G)$ which is defined as

$$W = \langle u, y \mid u^4 = y^4 = [u, y] = 1\rangle.$$

We also define $W_0=\Omega_1(W)$.

In addition, the generators u and y are chosen so that $\langle y, W_0\rangle \lhd G$.

1.5. *The structure of* $c_G(W)$. Let $C=c_G(W)$. Since G contains no normal E_8 and $W\leqq G$, we see [1] that $c_G(W)=C$ is metacyclic.

In addition we may apply 1.3 to C to show that C' has order at most 2. Thus, C is either a 2-generator abelian group or C is a metacyclic group with $|C'|=2$.

Suppose C is abelian. Then

$$C = \langle a, b \mid a^{2^\alpha} = b^{2^\beta} = [a, b] = 1, \alpha \geqq \beta \geqq 2\rangle.$$

Now if $\alpha>\beta+1$, then $\mho^\beta(C)=\langle a\rangle^{2^\beta}$ is a G-normal cyclic subgroup of order greater than 2, contradicting 1.3. So $\beta+1\geqq\alpha\geqq\beta\geqq 2$, and we have the abelian groups

$$C = C(n, n, 0) = \langle a, b \mid a^{2^n} = b^{2^n} = [a, b] = 1, n \geqq 2\rangle,$$
$$C = C(n+1, n, 0) = \langle a, b \mid a^{2^{n+1}} = b^{2^n} = [a, b] = 1, n \geqq 2\rangle.$$

Now we consider the case in which C is nonabelian. In this case, C/C' is a 2-generator abelian group. So we can find $a, b \in C$ such that $C/C' = \langle aC' \rangle \times \langle bC' \rangle$. Without loss of generality we may assume $|a| \geqq |b|$. If $\langle a \rangle \cap \langle b \rangle \neq 1$, then since $\langle a \rangle \cap \langle b \rangle \lhd G$ and cyclic it follows that $\langle a \rangle \cap \langle b \rangle = C'$. Let α, β be integers such that $a^{2^\alpha} = b^{2^\beta} = [a, b]$. Then $\alpha \geqq \beta \geqq 2$ since $a, b \notin W$ (otherwise C is abelian). Now let $b^* = a^{2^{\alpha-\beta}}b$. Since $\beta \geqq 2$, $(b^*)^{2^\beta} = (a^{2^{\alpha-\beta}}b)^{2^\beta} = a^{2^\alpha}b^{2^\beta} = 1$. So we may assume that C has the following defining relations:

$$C = \langle a, b \mid a^{2^\alpha} = b^{2^\beta} = 1, [a, b] = a^{2^{\alpha-1}}, \alpha \geqq \beta \geqq 2 \rangle.$$

Now we consider the values of α and β. If $\alpha \geqq \beta + 2$, then $\langle a^{2^{\alpha-2}} \rangle = \mho^{\alpha-2}(C)$ is a cyclic normal subgroup of G, contradicting 1.3. In addition $\beta \geqq 3$, since C is nonabelian.

Thus, if C is nonabelian, either

$$C = C(n, n, 1) = \langle a, b \mid a^{2^n} = b^{2^n} = 1, [a, b] = a^{2^{n-1}}, n > 2 \rangle$$

or

$$C = C(n+1, n, 1) = \langle a, b \mid a^{2^{n+1}} = b^{2^n} = 1, [a, b] = a^{2^n}, n > 2 \rangle.$$

1.6. *The involutions of* $\Phi(G)\backslash C$. Let v be an involution of $\Phi(G)\backslash C \cap \Phi(G)$. Then v acts on W as one of the automorphisms in $\Phi(A)$, cf. 1.2. $\langle v, W_0 \rangle$ is an E_8, so $\langle v, W_0 \rangle \ntriangleleft G$. Thus, there is a $g \in G$ such that $v^g \notin \langle v, W_0 \rangle$. Now the group $\langle v, v^g \rangle$ is dihedral so $(vv^g)^v = (vv^g)^{-1}$. Since $v \in \Phi(A) = Z(A)$, we see that v^g induces the same automorphism as v on W. Thus, $vv^g = v^{-1}v^g \in c_G(W) = C$. Since $vv^g \notin W_0$, the order of vv^g is greater than 2. Thus, since v inverts vv^g, we see that v inverts some element of $W \backslash W_0$.

The nonidentity elements of $\Phi(A)$ are η, $\eta\sigma^2$, and σ^2. The automorphism σ^2 inverts no element of $W\backslash W_0$. So v acts as η or $\eta\sigma^2$.

1.7. *If* $|C/W| \geqq 4$, *then no element of* G *induces the automorphism* $\eta\sigma^2$ *on* W.

Proof. Suppose $g \in G$ induces the automorphism $\eta\sigma^2$ on W. Then $g^2 \in C$, and since g fixes no element of $W\backslash W_0$, $g^2 \in W_0$. Now since $|C| \geqq 2^6$, C contains a G-normal subgroup C_0 of order 2^6 and C_0 has defining relations

$$C_0 = \langle a_0, b_0 \mid a_0^8 = b_0^8 = 1, [a_0, b_0] = a_0^{4\delta} \text{ where } \delta \in Z_2 \rangle.$$

Let $y = a_0^2$, $u = b_0^2$. Then

$$\begin{aligned} b_0^g &= b_0^{-1+4\alpha}a_0^{2+4\beta} && \text{where } \alpha, \beta \in Z_2, \\ a_0^g &= b_0^{4i}a_0^{-1+4j} && \text{where } i, j \in Z_2. \end{aligned}$$

Now

$$b_0 = b_0^{g^2} = (b_0^{-1+4\alpha}a_0^{2+4\beta})^g = b_0a_0^{-4}.$$

Since $a_0^4 \neq 1$, there is no element $g \in G$ which induces the automorphism $\eta\sigma^2$ on W if $|C/W| > 2$.

1.8. *If $C>W$, then no element of G induces the automorphism σ^2 on W.*

Proof. We first consider the case $|C/W|>2$. Suppose $s\in G$ induces the automorphism σ^2 on W. By 1.7 we see that G has no element which induces the automorphism $\eta\sigma^2$ on W. However, from 1.6 we see that G must have an element v which induces the automorphism η on W. But this means that vs induces the automorphism $\eta\sigma^2$ on W, a contradiction.

Thus, we must only consider the case $|C/W|=2$. In this case $C=\langle a, u \mid a^8=u^4=[a, u]=1\rangle$ and $a^2=y$. We again assume $s\in G$ induces the automorphism σ^2 on W. Hence, $s^2\in C$. Now by 1.6 we see that s is not an involution.

The action of s on C is given by $a^s=a^{1+4i}u^{2j}$ where $i, j\in Z_2$; $u^s=ua^4$, and $s^2\in c_C(s)\leqq\langle a, u^2\rangle$.

If $s^2\in\langle u^2, a^2\rangle$, then by replacing s by $s^*=sa^{\alpha}u^{\beta}$ for the appropriate choices of α, β we have $(s^*)^2=1$ which is contradictory to the fact that no involution of G induces the automorphism σ^2 on W.

Thus, $s^2=\langle a, u^2\rangle\backslash\langle a^2, u^2\rangle$. So let $s^2=a_0$. Then $C=\langle a_0, u \mid a_0^8=u^4=[a_0, u]=1\rangle$, and $a_0^s=a_0$ and $u^s=ua_0^4$.

Next we show that if $|C/W|=2$, G contains no element t which induces the automorphism σ on W. If t induces the automorphism σ on W, then $t^4=a$ and $u^t=ua^2$. Now G contains an involution v which induces the automorphism η or $\sigma^2\eta$ on W. Since $\eta\in Z(A_G(W))$, $t^v\equiv t \pmod C$. So $t^v=tu^ia^j$ for some integers i, j. Thus, $(t^2)^v=(tu^ia^j)^2=t^2u^{2i}a^{2(i+j)}$ and $a^v=(t^4)^v=(t^2u^{2i}a^{2(i+j)})^2=a^{1+4(i+j)}$. Thus, $a^v\equiv(t^4)^v\equiv a \pmod{W_0}$. But $a^v\equiv a^{-1} \pmod{W_0}$, a contradiction. Thus, if $|C/W|=2$, no element of G induces the automorphism σ on G.

Now we complete the proof. We consider the group G/C. Let $^-$ denote the natural homomorphism from G onto G/C. Since $\bar{s}$ is not a square in $\bar{G}$ and $\bar{v}\in\Phi(\bar{G})$, we see that $\bar{v}$ is a square in $\bar{G}$. Let $k\in G$ such that $\bar{k}^2=\bar{v}$. Thus, $s^k\equiv s \pmod C$ and so $s^k=su^ia^j$. Thus,

$$a^k = (s^2)^k = (su^ia^j)^2 = a^{1+2j+4i}u^{2i}.$$

So $a^{k^2}\equiv a^{(1+2j)^2}\equiv a \pmod{W_0}$. However, $a^v\equiv a^{-1} \pmod{W_0}$, a contradiction. Thus, $s\notin G$ if $C>W$.

1.9. *If $c_G(W)=W$ and $\sigma^2\in A_G(W)$, then G contains an element s which induces the automorphism σ^2 on W and $s^2=y$.*

Proof. Let s_0 be an element of G inducing the automorphism σ^2 on W and $s_0^2\neq y$. Since $s_0^2\in c_W(\sigma^2)=\langle u^2, y\rangle$, we see that s_0 is not an involution, for otherwise $\langle s_0, W_0\rangle$ is a G-normal E_8 (cf. 1.6). If $s_0^2\in W_0$, then letting $s_1=s_0u^iy^i$ for suitable i, j, we have $s_1^2=1$, so $s_0^2\in\langle y, u^2\rangle\backslash\langle u^2, y^2\rangle$. Thus, $s_0^2=y^iu^{2j}$, where $i=\pm 1$. Let $s=s_0^{i+2j}u^j$. Then $s^2=y$. Q.E.D.

We summarize the results of §1 in the following theorem.

THEOREM 1. *Let G be a finite 2-group in which $\Phi(G)$ contains an elementary abelian subgroup of order* 8, *but G contains no normal elementary abelian subgroup of order* 8.

Then

(i) $\Phi(G)$ *contains a G-normal subgroup* $W \simeq C_4 \times C_4$.

(ii) *The centralizer in G of W is isomorphic to one of the following groups:*

$$C(m, n, \varepsilon) = \langle a, b \mid a^{2^m} = b^{2^n} = 1, [a, b] = a^{\varepsilon 2^{m-1}} \rangle$$

where $n+1 \geqq m \geqq n \geqq 2$ *and* $\varepsilon \in Z_2$.

(iii) $\Phi(G)$ *contains an involution v which induces either the automorphism* η *or* $\eta\sigma^2$ *on W.*

(iv) *If* $c_G(W) > W$, *then no element of G induces the automorphism* σ^2 *on W.*

(v) *If* $|c_G(W)/W| > 2$, *then no element of G induces the automorphism* $\eta\sigma^2$ *on W.*

(vi) *If* $c_G(W) = W$ *and* $s \in G$ *induces the automorphism* σ^2 *on W, then* $s^2 = y$.

The structural conditions of Theorem 1 enable us to construct all (∗)-groups as extensions of one of the groups $C(m, n, \varepsilon)$.

2. **The (∗)-groups for which** $c_G(W) = W$.

2.1. *If G is a (∗)-group for which* $c_G(W) = W$, *then either G has an element k such that* k^2 *induces the automorphism* η *on W or G has an element r such that* r^2 *induces the automorphism* $\sigma^2\eta$ *on W.*

Proof. By Theorem 1, $\Phi(G)$ contains an element v which induces either the automorphism η or $\sigma^2\eta$ on W. If v is a square we are done. So assume v is not a square. Then $\Phi(G)/W$ is not cyclic and hence $\Phi(G)/W \simeq \langle \eta, \sigma^2 \rangle$. Hence G contains an element g such that g^2 induces one of the automorphisms η or $\sigma^2\eta$.

2.2. *If G is a (∗)-group, which contains an element k such that* k^2 *acts as* η *on W and* $c_G(W) = W$, *then one of the following groups is a proper subgroup of G:*

$$K = \langle k, u \mid k^4 = u^4 = 1, u^k = u^{-1}y, y^4 = [y, u] = 1, y^k = u^2 y \rangle,$$
$$\hat{K} = \langle k, u \mid k^8 = u^4 = 1, u^k = u^{-1}y, y^2 = k^4, y^k = u^2 y, [y, u] = 1 \rangle.$$

Let $\bar{k}$ be the automorphism induced by k on W. Then $\bar{k}^2 = \eta$ implies $\bar{k} \in \lambda\langle \eta, \sigma^2 \rangle$. Thus, $\langle \bar{k} \rangle = \langle \lambda \rangle$ or $\langle \bar{k} \rangle = \langle \lambda\sigma^2 \rangle$. If $\bar{k}$ acts as $\lambda\sigma^2$, then let $y_0 = y^{-1}$. Thus, $u^{\lambda\sigma^2} = u^{-1}y^{-1} = u^{-1}y_0$ and $y_0^{\lambda\sigma^2} = u^2 y_0$. So by suitable choice of generators of W we may assume that $u^k = u^{-1}y$ and $y^k = u^2 y$. Now $k^2 \in c_W(k) = \langle y^2 \rangle$. Hence, $k^4 = y^{2\alpha}$ for $\alpha \in Z_2$. So the group $\langle k, W \rangle$ is isomorphic to one of the groups K or $\hat{K}$ depending on whether $\alpha = 0$ or $\alpha = 1$. We note that K has a normal E_8, e.g., $\langle k^2, u^2, y^2 \rangle$ and $\hat{K}$ contain no E_8. So neither K nor $\hat{K}$ is a (∗)-group.

Note. We will show (§2.5.2) that $\hat{K}$ is not a subgroup of any (∗)-group for which $c_G(W) = W$.

2.3. *If G is a (∗)-group for which* $c_G(W) = W$ *and G contains an element r such that* r^2 *induces the automorphism* $\eta\sigma^2$ *on W, then G contains one of the following groups as a proper subgroup:*

$$R = \langle r, u \mid r^4 = u^4 = 1, u^r = uy, y^4 = [u, y] = 1, y^r = u^2 y \rangle,$$
$$\hat{R} = \langle r, u \mid r^8 = u^4 = 1, u^r = uy, y^2 = r^4, [u, y] = 1, y^r = u^2 y \rangle.$$

Proof. The automorphism induced by r on W lies in the coset $\lambda\rho\langle\sigma^2, \eta\rangle$ of $A/Z(A)$. By a suitable choice of generators of W (i.e., replacing y by $[u, r]$, if necessary) we may assume $u^r = uy$, $y^r = u^2y$. Now $r^4 \in c_W(r) = \langle y^2\rangle$. Thus, $r^4 = 1$ or $r^4 = y^2$.

If $r^4 = 1$, then the group $\langle r, W\rangle = R$, and if $r^4 = y^2$, the group $\langle r, W\rangle = \hat{R}$. Since $(r^2u^iy^j)^2 = r^4y^{2i}$ for all $i, j \in Z_4$, we see that the involutions of $R\backslash W_0$ are all elements of the form $r^2u^\alpha y^\beta$, for $\alpha \in Z_2$, $\beta \in Z_4$, and the involutions of $\hat{R}\backslash W_0$ are all elements of the form $r^2u^\alpha y^\beta$, where $\alpha \equiv 1 \pmod 2$ and $\alpha, \beta \in Z_4$.

Neither R nor $\hat{R}$ is a $(*)$-group since R contains a normal E_8, namely, $\langle r^2, W_0\rangle$, and $\Phi(\hat{R})$ contains no E_8.

2.4. *Plan of attack for* §2. From 2.1–2.3 we see that each $(*)$-group G for which $c_G(W) = W$ is an extension of one of the groups $K, \hat{K}, R$, or $\hat{R}$ by elements of Aut (W). We will construct all of these groups by forming cyclic extensions of the above groups and then cyclic extensions of these extensions, etc. We will denote an extension G of K ($\hat{K}$, R, or $\hat{R}$) of degree 2^i as an ith stage extension.

2.5. *The* 1*st stage extensions of* K *and* $\hat{K}$. If G is a 1st stage extension of K or $\hat{K}$, then since the index of K (or $\hat{K}$) in G is 2, $K \lhd G$ (or $\hat{K} \lhd G$). Hence $A_G(W) \leqq \langle\lambda, \sigma^2, \sigma\rho\rangle$. Thus $A_G(W)$ is one of the groups $\langle\lambda, \sigma^2\rangle$ or $\langle\lambda, \sigma^\delta\rho\rangle$ where $\delta = \pm 1$.

2.5.1. *The* $(*)$-*group* G *such that* $c_G(W) = W$ *and* $A_G(W) = \langle\lambda, \sigma^2\rangle$. If G is such a group, then since $\Phi(G)$ contains $\Phi(K)$ or $\Phi(\hat{K})$, and $\Phi(\hat{K})$ contains no E_8, we see that $\hat{K} \nleqq G$. Let $s \in G$ such that s induces the automorphism σ^2 on W. Then by 1.9 we may assume $s^2 = y$. Since $\lambda^{\sigma^2} = \lambda$ we can find $i, j \in Z_4$ so that $k^s = ku^iy^j$. Now $u^2 = [k, y] = [k, s^2] = [k, s][k, s]^s = u^{2i}y^{2(i+j)}$.

2.5.1.2. So $i \equiv j \equiv 1 \pmod 2$.

Now we show that the mapping induced by s on K is an automorphism.

(i) $(k^s)^2 = k^2u^{2j}y^{j+2i}$ and so $(k^s)^4 = k^4 = 1$.

(ii) $[k^s, u^s] = [ku^iy^j, uy^2] = u^2y = (u^s)^2y^s$.

(iii) $[k^s, y^s] = [ku^iy^j, y] = u^2 = (u^s)^2$.

Thus, s induces an automorphism on K so the possibilities for G are:

2.5.1.3. $S(i,j) = \langle k, s \mid k^4 = s^8 = 1, k^s = ks^{2j}u^i$ where $i \equiv j \equiv 1 \pmod 2$, $u^4 = 1$, $u^s = us^4$, $u^k = u^{-1}s^2\rangle$.

We now show that the groups $S(i,j)$ are $(*)$-groups. We find the involutions of $S = S(i,j)$. If $g \in S$ is an involution, then $g^2 \equiv 1 \pmod W$. The involutions of the abelian group S/W lie in the group $\langle v = k^2, s, w\rangle/W$. So $g = s^\alpha v^\beta w$ where α, β are integers and $w \in W$. Now since $(sw)^2 \equiv s^2 \equiv y \pmod{W_0}$, no element of the form sw for $w \in W$ is an involution. In addition $(sv)^2 \equiv u^2 \pmod{y^2}$ and $(svw)^2 \equiv (sv)^2 \equiv u^2 \not\equiv 1 \pmod{y^2}$. So no element of the form svw where $w \in W$ is an involution. Thus, the involutions of S all lie in the group $\langle v, W\rangle$.

Hence, if E is a G-normal E_8, then $E \leqq \langle k, W\rangle$. Thus, E is either the group $\langle v, W_0\rangle$ or $\langle vy, W_0\rangle$. Since $v^s \equiv vy \pmod{W_0}$, we see that S has no normal E_8, and so S is a $(*)$-group.

2.5.2. Lemma. *Let* G *be a* $(*)$-*group for which* $W = c_G(W)$ *and* $\lambda \in A_G(W)$. *If* $k \in G$ *induces the automorphism* λ *on* W, *then* $k^4 = 1$.

Proof. Suppose $k^4 \neq 1$. Then $(k^2w)^2 = k^4 = y^2$ for all $w \in W$. Hence $\Phi(G) > \langle k^2, W\rangle$, since $\Phi(G)$ contains an E_8. So we see that G contains an element s which acts as σ^2 on W and $s^2 = y$. In addition, $k^s = ku^iy^j$ where $i \equiv j \equiv 1 \pmod 2$. So if x is an involution of $\Phi(G)\backslash W$, $x = svu^my^n$ for some $m, n \in Z_4$, where $v = k^2$.

Now $(svw)^2 \equiv s^2v^2 \equiv y \pmod{W_0}$ for all $w \in W$. Thus, $\Phi(G)\backslash W$ contains no involution, which contradicts the assumption that G is a $(*)$-group.

2.5.3. *The $(*)$-groups G such that $c_G(W) = W$ and $A_G(W) = \langle \lambda, \sigma^\delta\rho\rangle$ where $\delta = \pm 1$.* Let G be such a group. Then, by 2.5.2, G contains the subgroup K (cf. 2.2). So G is an extension of K by an element g_δ such that g_δ induces the automorphism $\sigma^\delta\rho$ for $\delta = \pm 1$. So $u^{g_\delta} = u^{-1}y^\delta$ and $y^{g_\delta} = y$.

Proof. We begin by investigating the amalgamation of g_δ. Now $(g_\delta)^2 \in c_W(g_\delta) = \langle y\rangle$. If $g^2 = y^2$, then $(g_\delta y)^2 = 1$ and if $g_\delta^2 = y$, then $(g_1uy)^2 = 1$ and $(g_{-1}u)^2 = 1$, so without loss of generality we may assume $g_\delta^2 = 1$.

Now $k^{g_\delta} \equiv k^{-1} \pmod W$ so let $k^{g_1} = k^{-1}u^\alpha y^\beta$. Then $k = k^{g_1^2} = (k^{-1}u^\alpha y^\beta)^{g_1} = ku^{-2\beta}$. Thus, $\beta \equiv 0 \pmod 2$. So let $\alpha_1 \in Z_4$, $\beta_1 \in Z_2$ be chosen so that $k^{g_1} = k^{-1}u^{\alpha_1}y^{2\beta_1}$.

Now we consider k^{g-1}. Let $k^{g-1} = k^{-1}u^\alpha y^\beta$. Then $k = k^{g^2_{-1}} = (k^{-1}u^\alpha y^\beta)^{g-1} = ku^{2\beta}y^{2\alpha}$. Thus, $\beta \equiv \alpha \equiv 0 \pmod 2$. Hence we may choose $\alpha_{-1}, \beta_{-1} \in Z_2$ so that $k^{g_{-1}} = k^{-1}u^{2\alpha_{-1}}y^{2\beta_{-1}}$.

Now we check that the mappings induced by the elements g_δ are automorphisms.

(i) $(k^{g_\delta})^4 = (k^{-1}u^{\alpha_\delta}y^{\beta_\delta})^4 = 1$.

(ii) $[u^{g_\delta}, k^{g_\delta}] = [u^{-1}y^\delta, k^{-1}u^{\alpha_\delta}y^{\beta_\delta}] = u^2y^{-1} = (u^2y^2)y = (u^{g_\delta})^2y^{g_\delta}$.

(iii) $[y^{g_\delta}, k^{g_\delta}] = [y, k^{-1}u^{\alpha_\delta}y^{\beta_\delta}] = u^2y^2 = (u^{g_\delta})^2$.

So we may construct the groups $\langle g_\delta, K\rangle = H_\delta$.

Now we check to see which of the groups H_δ are $(*)$-groups.

Suppose E is a normal E_8 of H_δ. Without loss of generality, we may assume $W_0 \leqq E$. Thus, the group EW/W is a normal subgroup of order 2 of the dihedral group H_δ/W. Hence, $EW/W = \langle v, W\rangle/W$. Now $E \lhd \langle k, W\rangle$ implies that E is either the group $\langle v, W_0\rangle$ or the group $\langle vy, W_0\rangle$. Since

$$\begin{aligned} v^{g_\delta} = (k^2)^{g_\delta} &= (k^{-1}u^{\alpha_1}y^{2\beta_1})^2 = k^2y^{\alpha_1} \pmod{W_0}, \\ &= (k^{-1}u^{2\alpha_{-1}}y^{2\beta_{-1}})^2 = k^2 \pmod{W_0}, \end{aligned}$$

we see that H_{-1} is not a $(*)$-group for any choice of α_{-1}, β_{-1}, and H_1 is a $(*)$-group if $\alpha_1 \equiv 1 \pmod 2$.

Thus, we have

$$H_1 = \langle g, K \mid g^2 = 1, k^g = k^{-1}u, u^g = u^k = u^{-1}y, \\ u^4 = y^4 = [u, y] = 1, y^k = u^2y\rangle.$$

2.6. *The 2nd stage extensions of K.* In this section we consider the $(*)$-groups G for which $A_G(W) \geqq \langle \lambda, \sigma^2\rangle$. Since $A/\langle \sigma^2, \lambda\rangle$ is a four-group, A has three maximal subgroups properly containing $\langle \sigma^2, \lambda\rangle$, namely, $\langle \sigma, \lambda\rangle$, $\langle \rho, \sigma^2, \lambda\rangle$, and $\langle \sigma\rho, \sigma^2, \lambda\rangle$.

2.6.1. *The* $(*)$-*groups* G *for which* $c_G(W)=W$ *and* $A_G(W)=\langle\lambda,\sigma\rangle$. Let G be such a group. Then by 2.5.2, G contains an element k so that $k^4=1$, and k induces the automorphism λ on W. Thus, G contains one of the groups $S(i,j)$. So let $G=\langle t, S=S(i,j)\rangle$, where t induces the automorphism σ on W. We may assume $t^4=y$, for if $t^4=y^2$, then t^2y is an involution, contradicting 1.9. So we may assume $t^2=s$. Let $k^t=k^{-1}su^\alpha y^\beta$, where $\alpha,\beta\in Z_4$. Then for $i\equiv j\equiv 1 \pmod 2$, we have

$$ku^iy^j = k^s = k^{t^2} = (k^{-1}su^\alpha y^\beta)^t = (k^{-1}su^\alpha y^\beta)^{-1}s(uy)^\alpha y^\beta.$$

So $ku^iy^j=ku^iy^ju^{2(\alpha-\beta)}$. Thus, $\alpha\equiv\beta \pmod 2$.

Now we check the relations of S under the action of t.

(i) $(k^t)^{s^t}=(k^{-1}su^\alpha y^\beta)^s=(ku^iy^j)^{-1}s(uy^2)^\alpha y^\beta=k^{-1}su^{(\alpha+i)+2(j-i)}y^{\beta+i+j-2i+2\alpha}$ and $k^t(u^t)^i(y^t)^j=k^{-1}su^{\alpha+i}y^{\beta+i+j}$. So we must have

$$k^{-1}su^{(\alpha+i)+2(j-i)}y^{\beta+i+j-2i+2\alpha} = k^{-1}su^{\alpha+i}y^{\beta+i+j}.$$

Hence, $i\equiv j\equiv\alpha\equiv\beta\equiv 1 \pmod 2$.

(ii) Since $(k^t)^2\equiv(k^{-1}s)^2\equiv k^{-2}s^2\equiv k^2 \pmod W$, we see that $(k^t)^2=vw$ for some element $w\in W$. Thus, $(k^t)^4=(vw)^2=w^vw=1$.

(iii) $[u^t,k^t]=[uy,k^{-1}su^\alpha y^\beta]=u^{-2}y^{-1}=(uy)^2y=(u^t)^2y^t$.

(iv) $[y^t,k^t]=[y,k^{-1}su^\alpha y^\beta]=u^2y^2=(u^t)^2$.

Thus, the groups of this type are the groups $T(\alpha,\beta,i,j)$ where $T(\alpha,\beta,i,j)=\langle t,k \mid t^{16}=k^4=1,\ k^t=k^{-1}t^{2+4\beta}u^\alpha$ where $\alpha\equiv\beta\equiv 1 \pmod 2$, $u^4=1$, $k^{t^2}=kt^{4j}u^i$ where $i\equiv j\equiv 1 \pmod 2$, $u^t=ut^4$, $u^k=u^{-1}t^4\rangle$.

We show $T=T(\alpha,\beta,i,j)$ is a $(*)$-group. Since $\Phi(T)\geqq\Phi(S)$ and $\Phi(S)$ contains an E_8, $\Phi(T)$ contains an E_8. If E is a normal E_8 of T, we may assume $E>W_0$. Then EW/W is a normal subgroup of order 2 of T/W. Thus,

$$EW/W \leqq Z(T/W) = \langle s, k^2, W\rangle/W.$$

So $E\leqq S$. However, S has no normal E_8, so T has no normal E_8. Thus,

$$T = T(\alpha,\beta,i,j)$$

is a $(*)$-group.

2.6.2. *There is no* $(*)$-*group* G *such that* $W=c_G(W)$ *and* $A_G(W)=\langle\lambda,\sigma^2,\sigma\rho\rangle$.

If there were such a group G, then G would be an extension of a group S isomorphic to one of the groups $S(i,j)$ by an element g which would induce the automorphism $\sigma\rho$ on W. Since $[\sigma^2,\sigma\rho]=1$ and $\lambda^{\sigma\rho}=\lambda^{-1}$ we could find integers $m,n,\alpha,\beta\in Z_4$ so that $s^g=su^my^n$, $k^g=k^{-1}u^\alpha y^\beta$, and so that the relation $(k^g)^{s^g}=k^g(u^g)^i(y^g)^j$ for $i\equiv j\equiv 1 \pmod 2$ would hold. Since $[s^2,g]=1$, $m\equiv n\equiv 0 \pmod 2$.

However, $(k^g)^{s^g}=(k^{-1}u^my^n)^{su^iy^j}=k^{-1}u^{\alpha+i}y^{i+j+2\alpha+\beta+2m}$ and $k^g(u^g)^i(y^g)^j=k^{-1}u^{\alpha-i}\cdot y^{\beta+i+j}$. Thus equating the exponents of u we would have $\alpha+i\equiv\alpha-i \pmod 4$ which would contradict the condition 2.5.1.2 that $i\equiv 1 \pmod 2$. So there is no $(*)$-group for which $A_G(W)=\langle\lambda,\sigma^2,\sigma\rho\rangle$.

2.6.3. *There is no* (∗)*-group* G *such that* $c_G(W)=W$ *and* $A_G(W)=\langle\lambda, \sigma^2, \rho\rangle$.

If G were such a group, then G would be an extension of a group S isomorphic to one of the groups $S(i,j)$ for $i\equiv j\equiv 1 \pmod 2$ (cf. 2.5.1) by an element x which would induce the automorphism ρ on W.

Since $(\sigma^2)^\rho=\sigma^2$ and $\lambda^\rho=\lambda\sigma^2$ we may assume that

$$s^x = su^my^n, \quad m\equiv n\equiv 0 \pmod 2); \qquad k^x = ksu^py^q, \quad p,q\in Z_4.$$

Then the relation $k^s=ku^iy^j$ (for $i\equiv j\equiv 1 \pmod 2$) should hold under the action of x. So we would have $(k^x)^{s^x}=(k^x)(u^x)^i(y^x)^j$, i.e. $(ksu^py^q)^{su^my^n}=(ksu^py^q)(u^{-1})^iy^j$. So

$$ksu^{i+p}y^{2i+j+2p+q+2m} = ksu^{-i+p}y^{q+j},$$

i.e. $i+p\equiv -i+p \pmod 4$. Thus, $i\equiv 0 \pmod 2$, a contradiction. Hence, S cannot be extended by x.

2.7. Theorem. *The* (∗)*-groups* G *for which* $c_G(W)=W$ *and* $\lambda\in A_G(W)$ *are* $S(i,j)$, H_1, *and* $T(\alpha,\beta,i,j)$.

2.8. *The* 1*st stage extensions of* R *and* $\hat{R}$. As in 2.5 we see that if G is a 1st stage extension of R or $\hat{R}$, then R (or $\hat{R}$) is normal in G so $A_G(W)\leqq N_A(\lambda\rho)=\langle\lambda\rho, \sigma^2, \sigma\rho\rangle$. Therefore $A_G(W)$ is one of the groups $\langle\lambda\rho,\sigma^2\rangle$ or $\langle\lambda\rho,\sigma^\delta\rho\rangle$ for $\delta=\pm 1$.

2.8.1. *The* (∗)*-groups* G *such that* $c_G(W)=W$ *and* $A_G(W)=\langle\lambda\rho,\sigma^2\rangle$. Let G be such a group. Then either $G=\langle s, R\rangle$ or $G=\langle s,\hat{R}\rangle$ where s induces the automorphism σ^2 on W and $s^2=y$ (by 1.9). In exactly the same way as in 2.5.1 we see that $s^2=y$ implies that $r^s=ru^iy^j$ where $i\equiv j\equiv 1 \pmod 2$, and we have the groups

$$\tilde{S}(i,j) = \langle r,s \mid r^4 = s^8 = 1, r^s = rs^{2j}u^i \text{ where } i\equiv j\equiv 1 \pmod 2,$$
$$u^4 = 1, u^r = us^2, u^2 = us^4\rangle$$

and

$$\hat{S}(i,j) = \langle r,s \mid r^8 = 1, s^4 = r^4, r^s = rs^{2j}u^i \text{ where } i\equiv j\equiv 1 \pmod 2,$$
$$u^4 = 1, u^r = us^2, u^s = us^4\rangle.$$

It is easily seen that the groups $\tilde{S}(i,j)$ and $\hat{S}(i,j)$ are (∗)-groups.

2.8.2. *The* (∗)*-groups* G *such that* $c_{G(W)}=W$ *and* $A_G(W)=\langle\lambda\rho,\sigma^\delta\rho\rangle$ *for* $\delta=\pm 1$. Let G be such a group. Then G is an extension of either the group R or the group $\hat{R}$ by an element g_δ such that $r^{g_\delta}\equiv r^{-1} \pmod W$. By the same arguments used in 2.5.3, we see that g_δ can be chosen so that $g_\delta^2=1$ and g induces either the automorphism $\sigma^\delta\rho$ for $\delta=\pm 1$. Choose m_δ, n_δ so that $r^{g_\delta}=r^{-1}u^{m_\delta}y^{n_\delta}$. Since $r=r^{g_\delta^2}=(r^{-1}u^{m_\delta}y^{n_\delta})^{g_\delta}$ $=ru^{2(m_\delta+n_\delta)}y^{m_\delta(\delta-1)}$, we find that $n_\delta\equiv m_\delta \pmod 2$ and $m_\delta(\delta-1)\equiv 0 \pmod 4$. So $r^{g_1}=r^{-1}u^my^n$ where $m\equiv n \pmod 2$ and $r^{g_{-1}}=r^{-1}u^my^n$ where $m\equiv n\equiv 0 \pmod 2$.

Now we must show that the mappings induced on R and $\hat{R}$ by g_δ are automorphisms.

(i) $(r^{g_\delta})^2=r^{-2}u^{2n_\delta}y^{m_\delta}$ and $(r^{g_\delta})^4=r^4$.

(ii) $[u^{g_\delta}, r^{g_\delta}]=[u^{-1}y^\delta, r^{-1}]=y=y^{g_\delta}$.

(iii) $[y^{g_\delta}, r^{g_\delta}]=[y, r^{-1}]=u^2y^2=(u^{g_\delta})^2$.

Thus, we may construct the groups $\langle g_\delta, R\rangle$ and $\langle g_\delta, \hat{R}\rangle$ for $\delta=\pm 1$.

Now the Frattini subgroup of each of these groups is the group $\langle r^2, W\rangle$. Thus, the Frattini subgroups of the groups $\langle g_\delta, \hat{R}\rangle$ for $\delta=\pm 1$ contain E_8's. Since the involutions of $\Phi(g_\delta, \hat{R})\backslash W$ are of the form $r^2u^iy^j$ for $i\equiv 1 \pmod 2$, the groups $\langle g_\delta, \hat{R}\rangle$ contain no normal E_8 for any choices of m_δ or n_δ. So we see that the groups $\langle g_\delta, \hat{R}\rangle$ are $(*)$-groups.

The groups $\langle g_\delta, R\rangle$ for which $m_\delta\equiv 0 \pmod 2$ have a normal E_8, namely, $\langle r^2, W_0\rangle$. So the only ones of the groups $\langle g_\delta, R\rangle$ which are $(*)$-groups are the groups $\langle g_1, R\rangle$ where $m_1\equiv n_1\equiv 1 \pmod 2$.

Thus, we list the groups of this type:

$$\tilde{H}(i,j) = \langle g, r \mid r^4 = g^2 = 1, r^g = r^{-1}u^iy^j \text{ where } i \equiv j \equiv 1 \pmod 2,$$
$$u^4 = y^4 = [u,y] = 1, u^g = u^{-1}y, u^r = uy, y^r = u^2y\rangle,$$
$$\hat{H}(\delta, i, j) = \langle r, g \mid r^8 = g^2 = 1, r^g = r^{-1}u^iy^j \text{ where } i, j \in Z_4, y^2 = r^4,$$
$$u^4 = [u,y] = 1, u^r = uy, y^r = u^2y, u^g = u^{-1}y^\delta \text{ where } \delta = \pm 1\rangle.$$

2.9. *The 2nd stage extensions of R and $\hat{R}$.* In this section we consider the $(*)$-groups G for which $c_G(W)=W$ and $A_G(W)\geqq\langle\lambda\rho, \sigma^2\rangle$. All 2nd stage extensions of R and $\hat{R}$ will be found among these groups. Since $A/\langle\lambda\rho, \sigma^2\rangle$ is a four-group, there are three choices for $A_G(W)$, namely $\langle\lambda\rho, \sigma\rangle$, $\langle\lambda\rho, \sigma^2, \sigma\rho\rangle$, and $\langle\lambda\rho, \lambda\rangle$. Now in 2.6.3 we showed that there is no $(*)$-group G such that $c_G(W)=W$ and $A_G(W)=\langle\lambda, \sigma^2, \rho\rangle=\langle\lambda\rho, \lambda\rangle$. Hence, we need only consider groups G such that $A_G(W)=\langle\lambda\rho, \sigma\rangle$ or $A_G(W)=\langle\lambda\rho, \sigma^2, \sigma\rho\rangle$. Each of these groups can be constructed as an extension of degree 2 of one of the groups $\tilde{S}(i,j)$ or $\hat{S}(i,j)$ for $i\equiv j\equiv 1 \pmod 2$.

2.9.1. *The $(*)$-groups G for which $c_G(W)=W$ and $A_G(W)=\langle\lambda\rho, \sigma^2, \sigma\rho\rangle$.* Let G be such a group. Then G is an extension of one of the groups $\tilde{S}(i,j)$ or $\hat{S}(i,j)$ by an element g which induces the automorphism $\sigma\rho$ on W. As in 2.8.2 we see that G can be chosen so that $g^2=1$ and $r^g=r^{-1}u^my^n$ where $m\equiv n \pmod 2$. We see that $[s,g]\in\langle y^2\rangle$, for if $s^g=su^\alpha y^\beta$, then

$$s = s^{g^2} = (su^\alpha y^\beta)^g = sy^{2\beta+\alpha}$$

and

$$y = (s^2)^g = (su^\alpha y^\beta)^2 = y(u^{2\alpha}y^{2(\alpha+\beta)}).$$

So $2\beta+\alpha=0 \pmod 4$ and $\alpha=\beta=0 \pmod 2$. Hence $\alpha\equiv 0 \pmod 4$ and $\beta\equiv 0 \pmod 2$. So let $s^g=sy^{2\varepsilon}$ where $\varepsilon\in Z_2$.

We now must show that the mappings induced by g on the groups $\tilde{S}(i,j)$ and $\hat{S}(i,j)$ are automorphisms of S. Since g induces an automorphism on R and $\hat{R}$, cf. 2.8.2, we need only check the relation $r^s=ru^iy^j$ for $i\equiv j\equiv 1 \pmod 2$ under the action of g.

Now

$$(r^g)^{s^g} = (r^{-1}u^my^n)^s = r^{-1}u^{i-2j+m}y^{-i+j+2m+n}$$

and

$$r^g(u^g)^i(y^g)^j = r^{-1}u^{m-i}y^{n+i+j}.$$

So using the fact that $m \equiv n \pmod 2$ we get $n \equiv m \equiv i \equiv j \equiv 1 \pmod 2$. Therefore, the groups of this type are

$$\tilde{X}(m, n, i, j, \varepsilon) = \langle g, \tilde{S}(i, j) \mid g^2 = 1, r^g = r^{-1}u^m s^{2n} \text{ where } m \equiv n \equiv 1 \pmod 2,$$
$$s^g = s^{1+4\varepsilon} \text{ where } \varepsilon = \pm 1\rangle,$$
$$\hat{X}(m, n, i, j, \varepsilon) = \langle g, \hat{S}(i, j) \mid g^2 = 1, r^g = r^{-1}u^m s^{2n} \text{ where } m \equiv n \equiv 1 \pmod 2,$$
$$s^g = s^{1+4\varepsilon} \text{ where } \varepsilon = \pm 1\rangle.$$

If $G = \tilde{X}(m, n, i, j, \varepsilon)$ or $G = \hat{X}(m, n, i, j, \varepsilon)$, then G is a $(*)$-group. For if E were a normal E_8 of G, then we may assume $W_0 \leqq E$ and so $EW/W \leqq \langle s, r^2, W\rangle/W$. Thus, E is a normal E_8 of $\tilde{S}(i, j)$ (or $\hat{S}(i, j)$). Since $\tilde{S}(i, j)$ (and $\hat{S}(i, j)$) have no normal E_8's, $E \ntrianglelefteq G$.

2.9.2. *There is no $(*)$-group G for which $c_G(W) = W$ and $A_G(W) = \langle \lambda\rho, \sigma\rangle$.*

If G were such a group, then G would be an extension of degree 2 of a group S isomorphic to one of the groups $\tilde{S}(i, j)$ or $\hat{S}(i, j)$ for $i \equiv j \equiv 1 \pmod 2$.

Let $G = \langle t, S\rangle$ where t induces the automorphism σ on W. Without loss of generality we may assume $t^2 = s$. Since $(\lambda\rho)^\sigma = (\lambda\rho)^{-1}\sigma^2$ we may choose $\alpha, \beta \in Z_4$ so that $r^t = r^{-1}su^\alpha y^\beta$.

If we check the relation $r^s = ru^i y^j$ under the action of t we see that

$$(r^t)^{s^t} = (r^{-1}su^\alpha y^\beta)^s = r^{-1}su^{i-2j+\alpha}y^{i+j+2\alpha+\beta}$$

and

$$r^t(u^t)^i(y^t)^j = r^{-1}su^{\alpha+i}y^{\beta+i+j}.$$

So from the exponents of u we would have $i - 2j + \alpha \equiv \alpha + i \pmod 4$, i.e. $2j \equiv 0 \pmod 4$ which contradicts the assumption that $j \equiv 1 \pmod 2$. Hence, there is no $(*)$-group G such that $W = c_G(W)$ and $A_G(W) = \langle \lambda\rho, \sigma\rangle$.

2.10. Theorem. *The $(*)$-group for which $c_G(W) = W$ and $\lambda\rho \in A_G(W)$ are $\tilde{S}(i, j)$, $\hat{S}(i, j)$, $\tilde{H}(i, j)$, $\hat{H}(\delta, i, j)$, $\tilde{X}(m, n, i, j, \varepsilon)$, and $\hat{X}(m, n, i, j, \varepsilon)$.*

3. The $(*)$-groups for which $|c_G(W)/W| = 2$.

3.1. If G is a $(*)$-group for which $|c_G(W)/W| = 2$, then by 1.8 $\sigma^2 \notin A_G(W)$. Thus, $\Phi(A_G(W))$ is either the group $\langle \eta\rangle$ or the group $\langle \eta\sigma^2\rangle$. So, by suitable choice of generators for W, we can either find an element $k \in G$ which induces the automorphism λ on W (if $\Phi(A_G(W)) = \langle \eta\rangle$) or an element $r \in G$ which induces the automorphism $\lambda\rho$ on W (if $\Phi(A_G(W)) = \langle \eta\sigma^2\rangle$).

3.2. *If G is a $(*)$-group for which $|c_G(W)/W| = 2$ and $\langle \eta\rangle = \Phi(A_G(W))$, then G contains one of the following $(*)$-groups as a normal subgroup*

$$K_\alpha = \langle k, a \mid k^4 = a^8 = 1, a^k = au, u^4 = [a, u] = 1, u^k = u^{-1}a^{2+4\alpha} \text{ for } \alpha \in Z_2\rangle,$$
$$\hat{K} = \langle k, a \mid k^8 = 1, a^4 = k^4, a^k = au, u^4 = [a, u] = 1, u^k = u^{-1}a^2\rangle.$$

Proof. From 3.1 we see that there is $k \in G$ such that k induces the automorphism λ on W. Now from 1.5 we have $c_G(W)=\langle a, W \mid a^2=y, [a, W]=1\rangle$. Thus, $a^k = u^{1+2\alpha}a^{1+4\beta}$ for some integers α, β; $u^k=u^{-1}a^2$.

Let $u_0=u^{1+2\alpha}a^{4\beta}$. Then $a^k=au_0$ and $u_0^k=u_0^{-1}a^{2+4\alpha}$. In addition, $a^{k^2}=a^{3+4\alpha}$, $u^{k^2}=u^{-1}$. Let k_α be defined by

$$a^{k_\alpha} = au, \quad u^{k_\alpha} = u^{-1}a^{2+4\alpha} \quad \text{for } \alpha \in Z_2.$$

Now $k_\alpha^4 \leqq C_W(k_\alpha)=\langle a^4\rangle$ and $(k_\alpha^2u^ia^j)^2=k_\alpha^4a^{4j(1+\alpha)}$. Thus, $k_1^4=(k_1^2u^ia^j)^2$ for all $i \in Z_4, j \in Z_8$. So if $k_1^4=1$, all elements of $\langle k^2, C\rangle \backslash W$ are involutions, and if $k_1^4=a^4$, then there are no involutions in $\langle k, C\rangle\backslash W$.

Now $(k_0^2a^iu^j)^2=k_0^2a^{4j}$, so if $k_0^4=1$, then all elements of $\langle k_0^2, W\rangle\backslash W$ are involutions. If $k_0^4=a^4$, then the coset $(k_0^2a)W$ contains all involutions of $\langle k_0C\rangle\backslash W$.

So we may construct the following groups:

$$K_\alpha = \langle a, k \mid a^8 = k^4 = 1, a^{k_\alpha} = au, u^4 = [a, u] = 1, u^{k_\alpha} = u^{-1}a^{2+4\alpha} \text{ for } \alpha \in Z_2\rangle,$$

$$\hat{K}_\alpha = \langle a, k_\alpha \mid a^8 = 1, a^4 = k_\alpha^4, a^{k_\alpha} = au, u^4 = [a, u] = 1, u^{k_\alpha} = u^{-1}a^{2+4\alpha} \text{ for } \alpha \in Z_2\rangle.$$

Now the group $\hat{K}_1$ is not a (∗)-group since $\Phi(\hat{K}_1)$ contains no E_8. $\Phi(K_\alpha)$ contains the subgroup $\langle k_\alpha^2, W_0\rangle$ which is an E_8 and $\Phi(\hat{K}_0)$ contains $\langle k_\alpha^2a, W_0\rangle$ which is an E_8. Since $[k_\alpha^2u^ia^j, a]\equiv a^2 \pmod{W_0}$, we see that no E_8 of the groups K_α or $\hat{K}_0$ is normal. Thus, K_α, $\hat{K}_0$ are (∗)-groups for $\alpha \in Z_2$.

It remains to show that if G is a (∗)-group containing one of the groups K_α or $\hat{K}_0$, then K_α (or $\hat{K}_0$) is normal in G. Since $\sigma^2 \notin A_G(W)$, the group $\langle\lambda\rangle \lhd A_G(W)$. Thus, $K_\alpha \lhd G$ (or $\hat{K}_0 \lhd G$).

We note that for any (∗)-group G which contains one of the K_α or $\hat{K}_\alpha$, $\Phi(G) =\langle k_\alpha^2, W\rangle$ (since $\sigma^2 \notin A_G(W)$). Thus, since $\Phi(\hat{K}_1)$ contains no E_8, $\hat{K}_1$ is not a subgroup of any (∗)-group.

3.3. *The (∗)-groups G which properly contain K_0, K_1, or $\hat{K}_0$.* If G is such a group, then since $\sigma^2 \notin A_G(W)$, $A_G(W)$ is either the group $\langle\sigma\rho, \lambda\rangle$ or the group $\langle\sigma^{-1}\rho, \lambda\rangle$. Thus, G contains an element g_δ which induces the automorphism $\sigma^\delta\rho$ on W where $\delta=\pm 1$. So

$$a^{g_\delta} = a^{1+4i}u^{2j}, \qquad u^{g_\delta} = a^{2\delta}u^{-1}.$$

Now if $(g_\delta)^2 \in W$, then by the same argument used in 2.3, we see that we may choose g_δ so that $g_\delta^2=1$. If $g_\delta \in c_G(W)\backslash W$, then we may choose $a=g_\delta^2$, so in this case $a^{g_\delta}=a$. In either case $a=a^{g_\delta^2}=(a^{1+4i}u^{2j})^{g_\delta}=a^{1+4j}$. So $j\equiv 0 \pmod 2$, and, hence,

$$a^{g_\delta} = a^{1+4\beta}, \qquad u^{g_\delta} = a^{2\delta}u^{-1} \quad \text{for } \delta = \pm 1.$$

We let $k_\alpha^{g_\delta}=k_\alpha^{-1}u^ia^j$ where $i \in Z_4, j \in Z_8$, and i and j depend on δ.

We consider two cases.

(i) *The case $g_\delta^2=1$.* In this case

$$k_\alpha = k_\alpha^{g_\delta^2} = (k_\alpha^{-1}u^ia^j)^{g_\delta} = (k_\alpha^{-1}u^ia^j)^{-1}(u^{-1}a^{2\delta})^i(a^{1+4\beta})^j.$$

So $k_\alpha = k_\alpha u^{-j} a^{4j\beta + 2i(\delta-1-2\alpha)}$. Thus, $j \equiv 0 \pmod 4$ and $i(\delta-1-2\alpha) \equiv 0 \pmod 4$. So $k_\alpha^{g_\delta} = k_\alpha^{-1} u^i a^{4j}$ where i satisfies the conditions of Table 3.3.1, and $j \in Z_2$.

3.3.1. TABLE.

α	δ	i
0	1	Arbitrary
0	-1	$\equiv 0 \pmod 2$
1	1	$\equiv 0 \pmod 2$
1	-1	Arbitrary

We now show that the mappings induced on K_α by the g_δ are automorphisms.

(i) $(k_\alpha^{-1} u^i a^{4j})^4 = k_\alpha^4$ (see the calculation in 3.2).

(ii) $[u^{g_\delta}, k_\alpha^{g_\delta}] = [u^{-1}a^{2\delta}, k_\alpha^{-1}u^i a^{4j}] = u^2 a^{2+4\alpha+4\delta} = (u^{g_\delta})^2 (a^{g_\delta})^{2+4\alpha}$.

(iii) $[a^{g_\delta}, k_\alpha^{g_\delta}] = [a^{1+4\beta}, k_\alpha^{-1}u^i a^{4j}] = a^{2+4\alpha}u^{-1}$.

Now $u^{g_\delta} = a^{2\delta}u^{-1}$, so $2\delta \equiv 2 + 4\alpha \pmod 8$. Hence, $\delta \equiv 1 + 2\alpha \pmod 4$, i.e. $\alpha = 0$ implies $\delta = 1$, and $\alpha = 1$ implies $\delta = -1$. Thus, we have the following groups:

$$H_0 = \langle g_1, K_0 \mid g_1^2 = 1, k_0^{g_1} = k_0^{-1}u^i a^{4j}, i \in Z_4, j \in Z_2, u^{g_1} = a^2 u^{-1}, a^{g_1} = a^{1+4\beta} \text{ where } \beta \in Z_2 \rangle,$$

$$\hat{H}_0 = \langle g_1, \hat{K}_0 \mid g_1^2 = 1, k_0^{g_1} = k_0^{-1}u^i a^{4j} \text{ where } i \in Z_4, j \in Z_2, u^{g_1} = a^2 u^{-1}, a^{g_1} = a^{1+4\beta} \text{ where } \beta \in Z_2 \rangle,$$

and

$$H_1 = \langle g_{-1}, K_1 \mid g_{-1}^2 = 1, k_1^{g_{-1}} = k_1^{-1}u^i a^{4j} \text{ where } i \in Z_4, j \in Z_2, u^{g_{-1}} = u^{-1}a^{-2}, a^{g_1} = a^{1+4\beta} \text{ where } \beta \in Z_2 \rangle.$$

Since K_α and $\hat{K}_0$ are (∗)-groups, it is easily seen that H_0, H_1, and $\hat{H}_0$ are (∗)-groups.

(ii) Now we construct the groups for which $g_\delta^2 = a$. In this case,

$$a^{g_\delta} = a, \qquad u^{g_\delta} = u^{-1}a^{2\delta},$$

and

$$k_\alpha u^{-1} = k_\alpha^a = k_\alpha^{g_\delta^2} = (k_\alpha^{-1}u^i a^j)^{g_\delta} = k_\alpha u^j a^{2i(\delta-1-2\alpha)}.$$

So $j \equiv -1 \pmod 4$, $i(\delta-1-2\alpha) \equiv 0 \pmod 4$. So $k_\alpha^{g_\delta} = k_\alpha^{-1}u^i a^{-1+4j}$ where i satisfies the condition of Table 3.3.1.

We now check to see if the mappings induced by g_δ on K_α and $\hat{K}_0$ are automorphisms:

(i) $(k_\alpha^{g_\delta})^4 = (k_\alpha^{-1}u^i a^{-1+4j})^4 = k_\alpha^4$.

(ii) $[u^{g_\delta}, k_\alpha^{g_\delta}] = [u^{-1}a^{2\delta}, k_\alpha^{-1}u^i a^{1+4j}] = u^{-2\delta}a^{2+4\alpha+4\delta} = (u^2 a^{2+4\alpha})^{g_\delta}$.

(iii) $[a^{g_\delta}, k_\alpha^{g_\delta}] = [a, k_\alpha^{-1}u^i a^{-1+4j}] = a^{2+4\alpha}u^{-1}$ and $u^{g_\delta} = a^{2\delta}u^{-1}$.

Hence, $2\delta \equiv 2 + 4\alpha \pmod 8$. Thus, $\delta \equiv 1 + 2\alpha \pmod 4$. So we obtain the following groups:

$$H(a, K_\alpha) = \langle g, K_\alpha \mid g^2 = a, k_\alpha^g = k_\alpha^{-1}u^i a^{-1+4j} \text{ where } i \in Z_4, j \in Z_2, u^g = u^{-1}a^{2+4\alpha} \text{ where } \alpha \in Z_2 \rangle,$$

$$\hat{H}(a, \hat{K}) = \langle g, \hat{K} \rangle = \langle g, \hat{K} \mid g^2 = a, k^g = k^{-1}u^i a^{-1+4j} \text{ where } i \in Z_4, j \in Z_2, u^g = u^{-1}a^2 \rangle.$$

The groups $H(a, K_\alpha)$ and $\hat{H}(g, \hat{K})$ are (*)-groups.

3.4. *If G is a (*)-group for which $|c_G(W)\backslash W|=2$ and $A_G(W)=\langle\eta\sigma^2\rangle$, then G contains a normal (*)-group R_α or $\hat{R}_\alpha$ where*

$$R_\alpha = \langle r_\alpha, a \mid r_\alpha^4 = a^8 = 1, a^{r_\alpha} = ua, u^4 = [u, a] = 1, u^{r_\alpha} = ua^{2+4\alpha} \text{ for } \alpha \in Z_2\rangle,$$

$$\hat{R}_\alpha = \langle r_\alpha, a \mid r_\alpha^8 = 1, r_\alpha^4 = a^4, a^{r_\alpha} = au, u^4 = [u, a] = 1, u^{r_\alpha} = ua^{2+4\alpha} \text{ for } \alpha \in Z_2\rangle.$$

Proof. The proof of this theorem is essentially the same as the proof of 3.2. We find an element r_α which induces the automorphism $\rho\lambda$ on W and then by suitable choice of generators show that

$$a^{r_\alpha} = ua, \qquad u^{r_\alpha} = ua^{2+4\alpha} \quad \text{for } \alpha \in Z_2.$$

Now $(r_\alpha a^i u^j)^4 = r_\alpha^4$ and $(r_\alpha^2 a^i u^j)^2 = r_\alpha^4 a^{4(i+j+i\alpha)} u^{2i}$. As in 2.11 we see that even in the case $r_\alpha^4 = a^4$ we have involutions in the group $\langle r_\alpha, a, u\rangle$, namely, the elements of the coset $r^2 u\langle a^2, u^2\rangle$. So we have the groups

$$R_\alpha = \langle r_\alpha, a \mid r_\alpha^4 = a^8 = 1, a^{r_\alpha} = au, u^{r_\alpha} = ua^{2+4\alpha} \text{ for } \alpha \in Z_2, u^4 = [a, u] = 1\rangle,$$

$$\hat{R}_\alpha = \langle r_\alpha, a \mid r_\alpha^8 = 1, r_\alpha^4 = a^4, a^{r_\alpha} = au, u^{r_\alpha} = ua^{2+4\alpha} \text{ for } \alpha \in Z_2, u^4 = [a, u] = 1\rangle.$$

Now $(r_\alpha^2 u^m a^n)^a \equiv (r_\alpha^2 u^m a^n) a^2 \pmod{W_0}$. So no subgroup of the form $\langle r_\alpha^2 u^m a^m, W_0\rangle$ is normal in either R_α or $\hat{R}_\alpha$. Thus, neither the R_α nor $\hat{R}_\alpha$ has a normal E_8. Since $\Phi(R_\alpha)$ and $\Phi(\hat{R}_\alpha)$ contain E_8's for $\alpha \in Z_2$, we see that R_α and $\hat{R}_\alpha$ are (*)-groups for $\alpha \in Z_2$.

3.5. *There is no (*)-group which properly contains any of the groups R_α or $\hat{R}_\alpha$.*

Proof. Suppose G is such a group and R is a subgroup of G isomorphic to one of the group R_α or $\hat{R}_\alpha$. Then by 1.7 we see that $|c_G(W)/W|=2$. Since $\sigma^2 \notin A_G(W)$ (cf. 1.8), then as in 3.3, we see that $G=\langle R, g_\delta\rangle$ where g_δ induces the automorphism $\sigma^\delta\rho$, for $\delta=\pm 1$, on W. Thus, we have

$$r_\alpha^{g_\delta} = r_\alpha^{-1}c \quad \text{for some } c \in C,$$
$$u^{g_\delta} = u^{-1}a^{2\delta} \quad \text{for } \delta = \pm 1,$$
$$a^{g_\delta} = a^{1+4\beta} \quad \text{for } \beta \in Z_2.$$

Since g_δ induces an automorphism on R, the relation $a^r = au$ should hold. But

$$(a^{g_\delta})^{r_\alpha^{g_\delta}} = (a^{1+4\beta})^{r_\alpha^{-1}c} = (a^{-1+4\alpha}u)^{1+4\beta} = a^{-1+4(\alpha-\beta)}u$$

and

$$(au)^{g_\delta} = a^{1+2\delta+4\beta}u^{-1}.$$

However, $a^{1+2\delta+4\beta}u^{-1} \neq a^{-1+4(\alpha-\beta)}u$, so we have a contradiction. Q.E.D.

4. **The (*)-groups for which $|c_G(W)/W|>2$.**

4.1. If G is a (*)-group for which $|c_G(W)/W|>2$, then by 1.7 and 1.8 we see that $\Phi(A_G(W))=\langle\eta\rangle$, and by suitable choice of generators for W we may assume

$\lambda \in A_G(W)$. If $\langle\lambda\rangle < A_G(W)$, then $A_G(W) = \langle\lambda, \rho\sigma\rangle$ or $A_G(W) = \langle\lambda, \rho\sigma^{-1}\rangle$. We begin by discussing extensions of the group $c_G(W)$ by an element k which induces the automorphism λ on W. Let $|c_G(W)| = 2^c$. We consider the cases in which c is even or odd separately.

4.2. *The* $(*)$*-groups for which* $|c_G(W)| = 2^{2n}$, $n \geqq 3$, *and* $A_G(W) = \langle\lambda\rangle$. If G is such a group, then there is an element $k \in G$ such that k induces the automorphism λ on W. Let $\varepsilon \in Z_2$ and

$$c_G(W) = C_\varepsilon = \langle a, b \mid a^{2^n} = b^{2^n} = 1, [a, b] = a^{2^{n-1}\varepsilon}\rangle.$$

By a suitable choice of generators for C_ε we can show that the automorphism induced by k on C_ε is $b^k = b^{-1}a$, $a^k = b^{-2}aw$ where $w \in \langle a^{2^{n-1}}, b^{2^{n-1}}\rangle$. In addition $k^4 \in C_W(k) = \langle a^{2^{n-1}}\rangle$ so either $k^4 = 1$ or $k^4 = a^{2^{n-1}}$. Hence we may construct the groups

$$K(n, n, \varepsilon, w) = \langle b, k \mid k^4 = b^{2^n} = 1 \text{ where } n \geqq 3, b^k = b^{-1}a, a^{2^n} = 1, [a, b] = a^{\varepsilon 2^{n-1}} \text{ where } \varepsilon \in Z_2, a^k = b^{-2}aw \text{ where } w \in \langle a^{2^{n-1}}, b^{2^{n-1}}\rangle\rangle$$

and

$$\hat{K}(n, n, \varepsilon, w) = \langle b, k \mid b^{2^n} = k^8 = 1 \text{ where } n \geqq 3, b^k = b^{-1}a, a^{2^{n-1}} = k^4, [a, b] = a^{\varepsilon 2^{n-1}} \text{ where } \varepsilon \in Z_2, a^k = b^{-2}aw \text{ where } w \in \langle a^{2^{n-1}}, b^{2^{n-1}}\rangle\rangle.$$

None of these groups has a normal E_8, and all have an E_8 contained in their Frattini subgroup except $\hat{K}(n, n, 0, 1)$. So, with the exception of the group $\hat{K}(n, n, 0, 1)$, the groups $K(n, n, \varepsilon, w)$ and $\hat{K}(n, n, \varepsilon, w)$ are $(*)$-groups.

4.3. *The* $(*)$*-groups for which* $|c_G(W)| = 2^{2n+1}$, $n \geqq 3$, *and* $A_G(W) = \lambda$. Let G be such a group, then $c_G(W)$ is given by

$$c_G(W) = C_\varepsilon = \langle a, b \mid a^{2^{n+1}} = b^{2^n} = 1, [a, b] = a^{2^n} \text{ where } \varepsilon \in Z_2\rangle.$$

Let $k \in G$ induce the automorphism λ on W. By suitable choice of generators we can show $a^k = b^{-1}a$, $b^k = b^{-1}a^2w$ where $w \in \langle a^{2^n}, b^{2^{n-1}}\rangle$. Since $k^4 \in c_W(k) = \langle a^{2^n}\rangle$, we obtain the following groups:

$$K(n+1, n, \varepsilon, w) = \langle k, a \mid k^4 = a^{2^{n+1}} = 1 \text{ where } n \geqq 3, a^k = b^{-1}a, b^{2^n} = 1, [a, b] = a^{\varepsilon 2^n} \text{ where } \varepsilon \in Z_2, b^k = b^{-1}a^2w \text{ where } w \in \langle a^2, b^{2^{n-1}}\rangle\rangle$$

and

$$\hat{K}(n+1, n, \varepsilon, w) = \langle k, a \mid a^{2^{n+1}} = 1, n \geqq 3, k^4 = a^{2^n}, a^k = b^{-1}a, b^{2^n} = 1, [a, b] = a^{\varepsilon 2^n} \text{ where } \varepsilon \in Z_2, b^k = b^{-1}a^2w \text{ where } w \in \langle a^{2^n n}, b^{2^{n-1}}\rangle\rangle.$$

It is clear that the groups $K(n+1, n, \varepsilon, w)$ are $(*)$-groups and that those groups of the form $\hat{K}(n+1, n, \varepsilon, w)$ which contain an E_8 are also $(*)$-groups. Since $(k^2b^ma^p)^2 = k^4w^{m+p}c^m$, i.e. $(k^2b^ma^p)^2 = a^{2^n(1+m\varepsilon)}w^{m+p}$, we see that $\hat{K}(n+1, n, w)\backslash W_0$ contains an involution if and only if $1 = a^{2^n(1+m\varepsilon)}w^{m+p}$, i.e. for $\varepsilon \equiv 0 \pmod 2$ and $w = a^{2^n}$ or $\varepsilon \equiv 1 \pmod 2$ and any choice of w.

4.4. *The $(*)$-groups G for which $|c_G(W)/W| > 2$ and $A_G(W) > \langle\lambda\rangle$.* These groups may be constructed as extensions of the groups discussed in 4.2 and 4.3 by elements g_δ where g_δ induces the automorphism $\sigma^\delta\rho$ on W for $\delta = \pm 1$. In the groups for which $|c_G(W)| = 2^{2n}$, g_δ may be chosen so that $g_\delta^2 = 1$; in the groups for which $|c_G(W)| = 2^{2n+1}$, g_δ may be chosen so that either $g_\delta^2 = 1$ or $g_\delta^2 = a$. The construction of these extensions is completely analogous to the work of §§2.5.3 and 3.3. Since the details of the computation become quite cumbersome and the number of parameters become exceedingly large, we will not discuss the detailed structure of these groups.

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